CUSP AND UNDULATION INVARIANTS OF RATIONAL CURVES.*

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Introduction.

It would be useful to have a formal statement of a method by means of which the cusp and undulation conditions of a rational plane curve could be written down in their best forms. These invariants are expressible in terms of the three-rowed determinants of the matrix of coefficients of the parametric equations of the curve. Meyer† has shown that the cusp condition is of order $2(n-1)$ and the undulation condition of order $4(n-3)$ in these determinants but it would be difficult to actually write them out by using his methods. Also, by the use of perspective curves, introduced by Stahl,‡ it is possible to write out the cusp condition as a determinant of order $6(n-1)$ but the high order of this determinant makes the method of little practical use. It is the purpose of this paper to give a straightforward method of writing out these invariants for a rational curve of order n as determinants of orders $2(n-1)$ and $4(n-3)$ whose constituents are the three-rowed determinants mentioned above. These methods are valuable because they may be directly extended to find the corresponding singularities in higher dimensions. Other interesting facts that are brought out in the paper are the peculiar relation of the cusp and undulation conditions of the plane rational quintic which is a special case of a relation peculiar to all invariants of the plane rational quintic, and the generalization of these facts in higher dimensions.

The Undulation Invariants.

Let the parametric equations of the rational curve of order n in space of d dimensions which we call R_d^n be written

$$(1) \qquad x_i = a_i t^n + n b_i t^{n-1} + \frac{n(n-1)}{1-2} c_i t^{n-2} \cdots; \qquad i = 0, 1, 2, 3, \cdots d;$$

also, when symbolic expressions are used for the n-ics on the right side of (1) it is to be understood that (1) is written in the form

$$(2) \qquad x_0 = (\alpha t)^n, \quad x_1 = (\beta t)^n, \quad x_2 = (\gamma t)^n, \quad \cdots x_d = (\pi t)^n.$$

* Presented to the American Mathematical Society, February 24, 1912.

† Meyer, Mathematische Annalen, vol. 38 (1891), pp. 375–6.

‡ W. Stahl, Mathematische Annalen, vol. 38 (1891), pp. 561–85.

Beginning with the plane we observe that any line

$$(3) \qquad (\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0$$

cuts (2) in n points whose parameters are the roots of the binary n-ic

$$(4) \qquad \xi_0(\alpha t)^n + \xi_1(\beta t)^n + \xi_2(\gamma t)^n = 0.$$

There are $3n - 6$ lines which cut (2) in three consecutive parameters; these $(3n - 6)$ parameters are the flex parameters of the R_2^n, and are the values of t which occur as cubed factors in members of the system of binary n-ics (4). There is no member of (4) which contains a fourfold factor unless a certain relation exists among the coefficients of the parametric equations of the R_2^n. Translating the theory of partial derivatives into symbolic notation* we find that the condition for the system (4) to have a member which contains a fourfold factor is expressed by

$$(5) \qquad (\alpha t)^{n-3}(\beta t)^{n-3}(\gamma t)^{n-3} \begin{vmatrix} \alpha_1^3 & \alpha_1^2\alpha_2 & \alpha_1\alpha_2^2 & \alpha_2^3 \\ \beta_1^3 & \beta_1^2\beta_2 & \beta_1\beta_2^2 & \beta_2^3 \\ \gamma_1^3 & \gamma_1^2\gamma_2 & \gamma_1\gamma_2^2 & \gamma_2^3 \end{vmatrix} = 0;$$

that is, if such a value of t exists it must satisfy simultaneously the four $(3n - 9)$-ics (5) obtained by dropping out in succession one column in the matrix. The coefficients of these $(3n - 9)$-ics are expressible in terms of the three-rowed determinants of the matrix of coefficients of the parametric equations of the R_2^n. The condition for t to satisfy these four equations simultaneously can be obtained as a determinant of order $4(n - 3)$ in their coefficients. Several illustrations will make this clear. In as much as calculation is an essential feature of the paper it will be necessary to write out explicitly more than the usual number of expressions. Take the R_2^4 as an example. From (1) and (2) we have

$$(6) \qquad R_2^4 = x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i; \qquad i = 0, 1, 2;$$

and

$$(7) \qquad x_0 = (\alpha t)^4, \quad x_1 = (\beta t)^4, \quad x_2 = (\gamma t)^4.$$

For (7) the expression (5) becomes

$$(8) \qquad (\alpha t)(\beta t)(\gamma t) \begin{vmatrix} \alpha_1^3 & \alpha_1^2\alpha_2 & \alpha_1\alpha_2^2 & \alpha_2^3 \\ \beta_1^3 & \beta_1^2\beta_2 & \beta_1\beta_2^2 & \beta_2^3 \\ \gamma_1^3 & \gamma_1^2\gamma_2 & \gamma_1\gamma_2^2 & \gamma_2^3 \end{vmatrix} = 0.$$

Let us actually calculate the two cubics found by striking out the fourth and third columns in the matrix (8); the first of these we shall refer to as (9) and the second as (10). Further

* See Grace & Young, Algebra of Invariants, p. 9.

$$
\begin{aligned}
(\alpha t)(\beta t)(\gamma t) &= (\alpha_1 t + \alpha_2)(\beta_1 t + \beta_2)(\gamma_1 t + \gamma_2) \\
&= \alpha_1\beta_1\gamma_1 t^3 + (\alpha_1\beta_1\gamma_2 + \alpha_1\beta_2\gamma_1 + \alpha_2\beta_1\gamma_1)t^2 \\
&\qquad + (\alpha_1\beta_2\gamma_2 + \alpha_2\beta_1\gamma_2 + \alpha_2\beta_2\gamma_1)t + \alpha_2\beta_2\gamma_2.
\end{aligned} \tag{11}
$$

Substituting in (9) and (10) from (11) and then from (6) in (9) and (10) we have

$$
\begin{aligned}
&\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} t^3 + \left[\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} a_0 & b_0 & c_0 \\ b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right] t^2 \\
&+ \left[\begin{vmatrix} a_0 & b_0 & c_0 \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 \\ b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & d_0 \\ b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right] t + \begin{vmatrix} b_0 & c_0 & d_0 \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = 0,
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
&\begin{vmatrix} a_0 & b_0 & d_0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix} t^3 + \left[\begin{vmatrix} a_0 & b_0 & d_0 \\ a_1 & b_1 & d_1 \\ b_2 & c_2 & e_2 \end{vmatrix} + \begin{vmatrix} a_0 & b_0 & d_0 \\ b_1 & c_1 & e_1 \\ a_2 & b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & e_0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}\right] t^2 \\
&+ \left[\begin{vmatrix} a_0 & b_0 & d_0 \\ b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & e_0 \\ a_1 & b_1 & d_1 \\ b_2 & c_2 & e_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & e_0 \\ b_1 & c_1 & e_1 \\ a_2 & b_2 & d_2 \end{vmatrix}\right] t + \begin{vmatrix} b_0 & c_0 & e_0 \\ b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \end{vmatrix} = 0,
\end{aligned} \tag{13}
$$

which are the cubics (9) and (10) in expanded form; the coefficients of (12) and (13) are expressible in terms of the three-rowed determinants of the matrix of coefficients of (6); we obtain

$$|abc|t^3 + |abd|t^2 + |acd|t + |bcd| = 0, \tag{12'}$$

$$|abd|t^3 + [|abe| + |acd|]t^2 + [|ace| + |bcd|]t + |bce| = 0; \tag{13'}$$

the other two cubics of (8) can be written out without actual calculation because of their symmetry with respect to (13′) and (12′); they are

$$|acd|t^3 + [|ace| + |bcd|]t^2 + [|ade| + |bce|]t + bde = 0, \tag{14}$$

$$|bcd|t^3 + |bce|t^2 + |bde|t + |cde| = 0. \tag{15}$$

Any three of these equations (12′), (13′), (14), and (15) could be solved for t^3, t^2, and t, and these values substituted in the fourth yields the condition* for there to be a value of t which satisfies the four simultaneously; that is

* It is better to say that this yields the condition for a value of t which satisfies three of the equations to satisfy the fourth.

the determinant of the coefficients of the four equations (12′), (13′), (14), and (15) which is

$$
(16)\qquad \begin{vmatrix} |abc|, & |abd|, & |acd|, & |bcd| \\ |abd|, & |abe| + |acd|, & |ace| + |bcd|, & |bce| \\ |acd|, & |ace| + |bcd|, & |ade| + |bce|, & |bde| \\ |bcd|, & |bce|, & |bde|, & |cde| \end{vmatrix} = 0
$$

is the condition for the R_2^4 to have an undulation. This* invariant for the R_2^4 has been found before, but it is given here as an illustration of the general method.

The Special Case of the R_2^5.

From equations (1) and (2) the R_2^5 may be written

$$
(17)\qquad x_i = a_i t^5 + 5b_i t^4 + 10c_i t^3 + 10d_i t^2 + 5e_i t + t_i, \qquad i = 0, 1, 2,
$$

or

$$
(18)\qquad \begin{aligned} x_0 &= (\alpha t)^5 \\ x_1 &= (\beta t)^5 \\ x_2 &= (\gamma t)^5. \end{aligned}
$$

The matrix corresponding to (5) in this case is

$$
(19)\qquad (\alpha t)^2(\beta t)^2(\gamma t)^2 \begin{vmatrix} \alpha_1^3 & \alpha_1^2\alpha_2 & \alpha_1\alpha_2^2 & \alpha_2^3 \\ \beta_1^3 & \beta_1^2\beta_2 & \beta_1\beta_2^2 & \beta_2^3 \\ \gamma_1^3 & \gamma_1^2\gamma_2 & \gamma_1\gamma_2^2 & \gamma_2^3 \end{vmatrix} = 0.
$$

For convenience let the following system of abbreviations be used for the R_2^5:

$$
(20)\qquad \begin{array}{lllll} \alpha = |abc| & \beta = |abd| & \gamma = |abe| & \delta = |abf| & \lambda = |acd| \\ \alpha' = |def| & \beta' = |cef| & \gamma' = |bef| & \delta' = |aef| & \lambda' = |cdf| \\ \mu = |ace| & \nu = |acf| & \psi = |ade| & \varphi = |bcd| & \chi = |bce| \\ \mu' = |bdf| & \nu' = |adf| & \psi' = |bcf| & \varphi' = |cde| & \chi' = |bde|. \end{array}
$$

The four sextics (19) may now be put in the form

$$
(21)\qquad \begin{aligned} &\alpha t^6 + 2\beta t^5 + (\gamma + 3\lambda)t^4 + (2\mu + 4\varphi)t^3 + (\psi + 3\chi)t^2 + 2\chi' t + \varphi' = 0, \\ &\beta t^6 + 2(\gamma + \lambda)t^5 + (\delta + 4\mu + 3\varphi)t^4 + (2\nu + 2\psi + 6\chi)t^3 \\ &\qquad\qquad + (\nu' + 4\chi' + 3\psi')t^2 + 2(\varphi' + \mu')t + \lambda' = 0, \\ &\lambda t^6 + (2\varphi + \mu)t^5 + (\nu + 4\chi + 3\psi)t^4 + (2\nu' + 2\psi' + 6\chi')t^3 \\ &\qquad\qquad + (\delta' + 4\mu' + 3\varphi')t^2 + 2(\gamma' + \lambda')t + \beta' = 0, \\ &\phi t^6 + 2\chi t^5 + (\psi' + 3\chi')t^4 + (2\mu' + 4\varphi')t^3 + (\gamma' + 3\lambda')t^2 \\ &\qquad\qquad + 2\beta' t + \alpha' = 0. \end{aligned}
$$

* See Transactions of the American Mathematical Society, vol. 12, No. 3, p. 304.

If these four equations are multiplied by t the four equations so obtained together with the four of (21) form 8 septemics whose determinant equated to zero is the required condition. Hence the undulation of the R_2^5 of (17) is

$$
\begin{vmatrix}
\alpha & 2\beta & \gamma+3\lambda & 2\mu+4\varphi & \psi+3\chi & 2\chi' & \varphi' & 0 \\
\beta & 2(\gamma+\lambda) & \delta+4\mu+3\varphi & 2\nu+2\psi+6\chi & \nu'+4\chi'+3\psi' & 2(\varphi'+\mu') & \lambda' & 0 \\
\lambda & 2(\varphi+\mu) & \nu+4\chi+3\psi & 2\nu'+2\psi'+6\chi' & \delta'+4\mu'+3\varphi' & 2(\gamma'+\lambda') & \beta' & 0 \\
\varphi & 2\chi & \psi'+3\chi' & 2\mu'+4\varphi' & \gamma'+3\lambda' & 2\beta' & \alpha' & 0' \\
0 & \alpha & 2\beta & \gamma+3\lambda & 2\mu+4\varphi & \psi+3\chi & 2\chi' & \varphi' \\
0 & \beta & 2(\gamma+\lambda) & \delta+4\mu+3\varphi & 2\nu+2\psi+6\chi & \nu'+4\chi'+3\psi' & 2(\varphi'+\mu') & \lambda' \\
0 & \lambda & 2(\varphi+\mu) & \nu+4\chi+3\psi & 2\nu'+2\psi'+6\chi' & \delta'+4\mu'+3\varphi' & 2(\gamma'+\lambda') & \beta' \\
0 & \varphi & 2\chi & \psi'+3\chi' & 2\mu'+4\varphi' & \gamma'+3\lambda' & 2\beta' & \alpha'
\end{vmatrix} = 0. \tag{22}
$$

So far as we know from the actual work the conditions (16) and (22) are only *necessary* conditions for undulations. Hence the question arises, do we obtain by this method a condition which is *sufficient* for an undulation? This question can be answered in the affirmative because in this way an invariant of order $4(n - 3)$ of the R^n is obtained which is not an identity and which vanishes when the R^n possesses this singularity. Since it is known* that an undulation of the R^n is conditioned by the vanishing of an invariant of order $4(n - 3)$ it is evident that the above method yields an invariant whose vanishing actually conditions an undulation. The same kind of argument may be used to show the sufficiency of the cusp conditions which are to be found in the following pages, and methods of this sort have been used by all writers† on invariants.

* See Brill, Mathematische Annalen, vol. 12, pp. 107–112; also, Meyer, loc. cit., p. 11.

† Salmon, Higher Algebra, Fourth Edition, p. 190–191; also, W. Stahl, loc. cit., p. 11.

It is necessary in this place to state several important theorems* regarding rational curves and to explain exactly what they mean when applied to the R_2^5. Every rational curve of order n in the plane is symmetrically represented by $n - 2$ binary forms of degree n; all line sections of the R_2^n are sets of points whose parameters are apolar to each of these $(n - 2)$ binary n-ics and therefore apolar to any linear combination of them. The combinants of these $(n - 2)$ binary n-ics are invariants of the R^n and are expressible in terms of the three-rowed determinants of the matrix of coefficients of the parametric equations of the curve by means of a scheme which we shall soon illustrate; also, the combinants of the three binary n-ics in the parametric equations of the curve are invariants of the R^n. When $n = 5$ or if we are considering the R_2^5 a very special relation exists: the R_2^5 is symmetrically represented by three binary quintics other than those which occur in the parametric equations of the R_2^5; the combinants of the three binary quintics in the parametric equations of the R_2^5 are invariants of the R_2^5; also, the general theory states that the combinants of the three binary quintics which are apolar to all line sections of the R_2^5 are invariants of the R_2^5. Let the three† binary quintics which are apolar to all line sections of the R_2^5 be

$$\begin{aligned}
&\alpha_0 t^5 + \alpha_1 t^4 + \alpha_2 t^3 + \alpha_3 t^2 + \alpha_4 t + \alpha_5 = 0,\\
(24)\qquad &\beta_0 t^5 + \beta_1 t^4 + \beta_2 t^3 + \beta_3 t^2 + \beta_4 t + \beta_5 = 0,\\
&\gamma_0 t^5 + \gamma_1 t^4 + \gamma_2 t^3 + \gamma_3 t^2 + \gamma_4 t + \gamma_5 = 0.
\end{aligned}$$

If the combinants of any three binary quintics are given the combinants of the three binary quintics (24) *in terms of the coefficients of* (17) *can be obtained by an easy substitution*‡ which we shall now give. The combinants of (24) are expressible in terms of the three-rowed determinants of the matrix

$$(25)\qquad \begin{Vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \end{Vmatrix}.$$

The matrix of coefficients of (17) is

$$(26)\qquad \begin{Vmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{Vmatrix}.$$

* Meyer's Apolarität und rationale Curven contains all these theorems; the expressions "symmetrically represented" and "fundamental involution" are used in this paper as he uses them.

† In (24) the binary quintics are purposely written without binomial coefficients.

‡ Meyer, Apolarität und rationale Curven, pp. 1–10.

Suppose the combinants of (24) are given in terms of the three-rowed determinants of the matrix (25); to obtain the combinants of the three binary quintics which symmetrically represent the R_2^5 *in terms of the three-rowed determinants of* (26), it is only necessary to make the substitution of complimentary determinants, i. e., for $|\alpha_0\, \beta_1\, \gamma_2|$ substitute $|def|$, for $|\alpha_2\, \beta_3\, \gamma_5|$ substitute $|abe|$, etc., after which allowance can be made for binomial coefficients, the exact substitution being illustrated in what follows. Since this method holds for any three binary quintics it must hold in particular for the binary quintics in (17). If an invariant of the R_2^5 is given in terms of the determinants of the matrix (26) *it may be looked upon as a combinant of the three binary forms* (17); by substituting $|def|$ for $|abc|$, $|bce|$ for $|adf|$, etc., we obtain the same combinant of the three binary quintics which symmetrically represent the R_5^2 *written without binomial coefficients*; hence by allowing for these coefficients the same combinants of the three binary quintics which symmetrically represent the R_2^5 of (17) are obtained. The exact substitutions which must be made in a combinant of the three binary quintics which occur in the parametric equations of the R_2^5 of (17) in order to obtain the same combinant of the three binary quintics which symmetrically represent the R_2^5 of (17) are in terms of the Greek letters of (20)

$$
(27)\qquad
\begin{array}{lllll}
\alpha = 50\alpha', & \beta = 50\beta', & \gamma = 100\lambda', & \delta = 500\varphi, & \lambda = 25\gamma', \\
\alpha' = 50\alpha, & \beta' = 50\beta, & \gamma' = 100\lambda, & \delta' = 500\varphi', & \lambda' = 25\gamma, \\
\mu = 50\mu', & \nu = 250\chi', & \psi = 50\psi', & \varphi = 5\delta', & \chi = 10\nu', \\
\mu' = 50\mu, & \nu' = 250\chi, & \psi' = 50\psi, & \varphi' = 5\delta, & \chi' = 10\nu.
\end{array}
$$

It might be supposed that each invariant of the R_2^5 by means of the substitutions (27) transforms into a multiple of itself, but this is not always the case. Consider the set of five points cut out of the R_2^5 by the covariant line* of the R_2^5 whose parameters are given by

$$
(28)\quad (2\lambda - \gamma)t^5 + (10\varphi - \delta)t^4 + (10\chi - 2\nu)t^3 + (10\chi' - 2\nu')t^2 \\
+ (10\varphi' - \delta')t + (2\lambda' - \gamma') = 0.
$$

By means of the substitutions (27) the equation (28) transforms into a quintic whose roots are the reciprocals of the roots of (28); hence the invariants of (28) are the same as the invariants of the transformed equation. The same is true of the 9-ic

* The equation of this line is $|afx| - 5|bex| + 10cdx = 0$ and is obtained from the apolarity condition of two line sections by making x_0, x_1, x_2 the coordinates of the point in which the two secant lines intersect.

$$(\alpha t)^3(\beta t)^3(\gamma t)^3 \begin{vmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2 \\ \beta_1^2 & \beta_1\beta_2 & \beta_2^2 \\ \gamma_1^2 & \gamma_1\gamma_2 & \gamma_2^2 \end{vmatrix} = 0 \tag{29}$$

which yields the 9 flex parameters* of the R_2^5 of (17). But if an invariant of (28) or (29) breaks up into two factors of the same degree it is by no means true that each factor transforms into itself, for each might transform into the other. In fact, this is exactly what does happen for the invariant (22). Each invariant of (29) as a whole must transform into itself by means of (27); the discriminant of (29) breaks up into two factors because it is known that two flexes of a rational curve unite in one way to form an undulation and in another to form a cusp; in the case of the R_2^5 these invariants are of the same degree and it is easy to verify that (22) does not transform into itself by means of (27); hence the transform of (22) by (27) yields the cusp invariant of the R_2^5 of (17). That this fact is a special case of a much more general theorem in regard to plane rational curves will appear when we consider the general cusp condition for rational curves. The cusp condition of the R_2^5 is found by making the substitutions (27) in the expression (22).

The Undulation Invariant of the R_2^6.

If we make $n = 6$ in (1), (2) and (5) and proceed as before the undulation condition of the R_2^6 is found by requiring four 9-ics to be satisfied by the same value of t.

The four 9-ics in expanded form are

$$\begin{aligned} &|abc|t^9 + 3|abd|t^8 + [3|abe| + 6|acd|]t^7 + [|abf| + 8|ace| + 10|bcd|]t^6 \\ &\quad + [3|acf| + 6|ade| + 15|bce|]t^5 + [3|adf| + 6|bcf| + 15|bde|]t^4 \\ &\quad + [|aef| + 8|bdf| + 10|cde|]t^3 + [3|bef| + 6|cdf|]t^2 + 3|cef|t \\ &\quad + |def| = 0 \end{aligned} \tag{34}$$

and

$$\begin{aligned} &|abd|t^9 + 3[abf + |acd|]t^8 + [3|abf| + 6|bcd| + 9|ace|]t^7 \\ &\quad + [|abg| + 8|ade| + 9|acf| + 18|bce|]t^6 \\ &\quad + [3|acg| + 9|adf| + 18|bcf| + 18|bde|]t^5 \\ &\quad + [3|adg| + 3|aef| + 6|bcg| + 15|cde| + 24|bdf|]t^4 \\ &\quad + [|abg| + 7|bdg| + |aeg| + 9|bef| + 8|cdf|]t^3 \\ &\quad + [3|bcg| + 6|cdg| + 9|cef|]t^2 + 3[ceg + |def|]t + |deg| = 0; \end{aligned} \tag{35}$$

* The expanded form of (29) is exactly the same as equation (34) on page 151.

the third 9-ic is the symmetrically formed 9-ic obtained from (35), its first coefficient being $|acd|$ and its last $|dfg|$; the fourth can be obtained as a 9-ic symmetrical with (34). By multiplying these four 9-ics by t^2 and t we obtain 8 equations which taken with the original 4 may be considered 12 11-ics whose determinant equated to zero is the undulation condition of the R_2^6. It is to be observed that (34) is the equation of the 9 flexes of the R_2^5 of (17); the corresponding expression for the R_2^7 is the flex 12-ic of the R_2^6, etc. Hence the problem of finding the $4(n-3)$-rowed determinant whose vanishing is the condition for the R_2^n to have an undulation is reduced to finding one new $(3n-9)$-ic—the correspondent of (35)—provided that the flex equation of the R_2^{n-1} is known.

The Cusp Invariant of R_2^n.

We recall that every R_d^n is symmetrically* represented by $n-d$ binary forms of order n and that combinants of these $n-d$ binary forms are invariants of the R_d^n. Hence in the plane the R_2^n is symmetricaly represented by $n-2$ binary forms of order n; i. e., the R_2^3 is symmetrically represented by a binary cubic whose invariants are invariants of the R_2^3 and since a binary cubic has only one invariant, its discriminant, the condition for the R_2^3 to have a cusp is the vanishing of this invariant; it is known that the condition for the R_2^4 to have a cusp is the condition for there to be a member of the pencil of binary quartics which symmetrically represent the R_2^4 which has a cubed factor; it has just been proved in the previous section of this paper that the condition for the R_2^5 to have a cusp is the condition for there to be a member of the system of binary quintics associated with the R_2^5 which contains a fourfold factor. Hence, the natural thing to suppose is that the R_2^n will have a cusp if only the associated system of $n-2$ binary forms of order n have a member which contains an $(n-1)$-fold factor. Is this true, or not?

That it is only one condition for there to be a member of a system of $n-2$ binary forms which contains an $(n-1)$-fold factor has been sufficiently illustrated in the previous paragraphs. Suppose this condition is imposed and that there is a member of this system of $n-2$ binary n-ics which contains an $(n-1)$-fold factor which may be taken ∞ as well as anything else. This member may be written in the form

$$\alpha_{n-1}t + \alpha_n = 0; \tag{36}$$

this n-ic (36) must be apolar to all line sections of the R_2^n of (1) which requires

$$\alpha_n a_i - \alpha_{n-1} b_i = 0; \qquad i = 0, 1, 2. \tag{37}$$

* Meyer, Apolarität und rationale Curven, p. 9.

Therefore

$$\frac{a_0}{b_0} = \frac{a_1}{b_1} = \frac{a_2}{b_2}. \tag{38}$$

What effect does this have upon the parametric equations of the R_2^n? By reason of (38) these may be written in the form

$$\begin{aligned} x_0 &= a_0 t^n + nb_0 t^{n-1} + \frac{n(n-1)}{2} c_0 t^{n-2} \cdots, \\ x_1 &= ka_0 t^n + knb_0 t^{n-1} + \frac{n(n-1)}{2} c_1 t^{n-2} \cdots, \\ x_2 &= qa_0 t^n + qnb_0 t^{n-1} + \frac{n(n-1)}{2} c_2 t^{n-2} \cdots. \end{aligned} \tag{39}$$

By choosing another triangle of reference $x_0 x_1' x_2'$, where $x_2' = x_2 - qx_0$ and $x_1' = x_1 - kx_0$, we may write (39) in the form

$$\begin{aligned} x_0 &= a_0 t^n + nb_0 t^{n-1} + c_0' t^{n-2} \cdots, \\ x_1' &= \qquad\qquad\qquad\;\; c_1' t^{n-2} \cdots, \\ x_2' &= \qquad\qquad\qquad\;\; c_2' t^{n-2} \cdots. \end{aligned} \tag{40}$$

Again taking as a new triangle of reference $x_0 x_1' x_2''$ where $x_2'' = c_2' x_1' - c_1' x_2'$ we may write the R_2^n

$$\begin{aligned} x_0 &= a_0 t^n + nb_0 t^{n-1} + c_0' t^{n-2} + d_0' t^{n-3} \cdots, \\ x_1' &= \qquad\qquad\qquad\;\; c_1' t^{n-2} + d_1' t^{n-3} \cdots, \\ x_2'' &= \qquad\qquad\qquad\qquad\qquad\;\; d_2' t^{n-3} \cdots. \end{aligned} \tag{41}$$

which shows that the R_2^n has a cusp, the cusp tangent being $x_2'' = 0$.

Hence, the condition for the R_2^n to have a cusp is the condition for there to be a set of its fundamental involution which contains an $(n-1)$-fold factor.*

The same method that has been used to write out the cusp invariants of the R_2^3, R_2^4, and R_2^5 can be applied generally. Let the $n-2$ binary n-ics whose linear system constitutes the fundamental involution of the R_2^n be written symbolically.

$$\left.\begin{aligned} (\alpha' t)^n &= 0, \\ (\beta' t)^n &= 0, \\ (\gamma' t)^n &= 0, \\ &\cdots\cdots \\ &\cdots\cdots \\ &\cdots\cdots \\ (\pi' t)^n &= 0 \end{aligned}\right\} \quad (n-2) \text{ equations.} \tag{42}$$

* The sufficiency of this condition is proved by a course of reasoning similar to that on page 8.

The condition for there to be a member of the linear system of (42) which contains an $(n-1)$-fold factor is the vanishing of the matrix

$$(43)\qquad (\alpha' t)^2(\beta' t)^2(\gamma' t)^2 \cdots (\pi' t)^2 \begin{vmatrix} {\alpha_1'}^{n-2} & {\alpha_1'}^{n-3}\alpha_2' & \cdots & {\alpha_2'}^{n-2} \\ {\beta_1'}^{n-2} & {\beta_1'}^{n-3}\beta_2' & \cdots & {\beta_2'}^{n-2} \\ {\gamma_1'}^{n-2} & {\gamma_1'}^{n-3}\gamma_2' & \cdots & {\gamma_2'}^{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ {\pi_1'}^{n-2} & {\pi_1'}^{n-3}\pi_2' & \cdots & {\pi_2'}^{n-2} \end{vmatrix}.$$

The matrix (43) yields $(n-1)$ equations of degree $(2n-4)$ which such a value of t must satisfy; the coefficients of these $(2n-4)$-ics are expressible in terms of the $(n-2)$-rowed determinants of the matrix of coefficients of (42): each of these $(n-1)$ equations has $(2n-3)$ coefficients; if these $(n-1)$ equations are multiplied by t we obtain $(n-1)$ other equations which taken with the original $(n-1)$ may be looked upon as $(2n-2)$ binary $(2n-3)$-ics; the determinant of these $(2n-2)$ equations is of order $2(n-1)$ in the $(n-2)$-rowed determinants of the matrix of coefficients of (42). The $(n-2)$-rowed determinants of this determinant are now replaced by three-rowed determinants of the matrix of coefficients of (1) when $n = n$ and $d = 2$ in this way; suppose a given $(n-2)$-rowed determinant is formed from the one matrix by striking out a certain set of three columns, the determinant to be substituted for it is formed from the other matrix—the matrix of coefficients of (1) for $n = n$ and $d = 2$—by using the constituents of these three columns. In this way it is possible to obtain the cusp condition of an R_2^n as the vanishing of a determinant of order $2(n-1)$ whose constituents are the three-rowed determinants of the type $|abc|$. If we understand that in both matrices binomial coefficients have been ignored the invariant obtained in the above maner would be that invariant for the R_2^n whose parametric equations do not contain binomial coefficients; hence allowance must be made for these in every case if we wish to find the cusp invariant for the R_2^n as written in (1). In the case of the R_2^6 suppose the four sextics of (42) are

$$(44)\qquad (\alpha' t)^6 = 0, \quad (\beta' t)^6 = 0, \quad (\gamma' t)^6 = 0, \quad (\delta' t)^6 = 0.$$

The cusp condition is imposed if there is a value of t which satisfies the matrix

$$(45)\qquad (\alpha' t)^2(\beta' t)^2(\gamma' t)^2(\delta' t)^2 \begin{vmatrix} \alpha_1'^4 & \alpha_1'^3\alpha_2' & \alpha_1'^2\alpha_2'^2 & \alpha_1'\alpha_2'^3 & \alpha_2'^4 \\ \beta_1'^4 & \beta_1'^3\beta_2' & \beta_1'^2\beta_2'^2 & \beta_1'\beta_2'^3 & \beta_2'^4 \\ \gamma_1'^4 & \gamma_1'^3\gamma_2' & \gamma_1'^2\gamma_2'^2 & \gamma_1'\gamma_2'^3 & \gamma_2'^4 \\ \delta_1'^4 & \delta_1'^3\delta_2' & \delta_1'^2\delta_2'^2 & \delta_1'\delta_2'^3 & \delta_2'^4 \end{vmatrix} = 0.$$

If (45) is expanded the result is 5 octavics whose coefficients are four-rowed determinants of the matrix of coefficients of (44); if the coefficients of these octavics are replaced by the three-rowed determinants of the matrix of coefficients of the parametric equations of the R_2^6 in the manner just explained, we have 5 octavics which multiplied by t furnish 5 other equations; these ten equations may be considered 10 9-ics, the determinant of whose coefficients equated to zero is the cusp condition of the R_2^6.

Higher Dimensions.

The methods used in the preceding paragraphs for rational curves in the plane can be applied without change of argument to rational curves in higher dimensions. For instance, in ordinary space an R_3^n has $4(n-3)$ *tetratactic* planes, or planes cutting the curve in four consecutive parameters; hence, by applying the method used to find the undulation condition in the plane we find the invariant which vanishes when the R_3^n has a pentatactic plane, and it can be expressed as a determinant of order $5(n-4)$ whose constituents are the four-rowed determinants of the matrix of coefficients of the parametric equations of the curve; also the singularity in higher spaces which corresponds to the cusp in the plane can be found in an analogous way. Also, as every invariant of the R_2^5 by means of (27) transforms into itself or into another invariant of the R_2^5, so any invariant of an R_k^{2k+1} transforms into itself or into another invariant of the R_k^{2k+1} by means of a scheme analogous to (27).

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